

Quantum Coherent String States in AdS_3 and $SL(2, R)$ WZWN Model

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Abstract

In this paper we make the connection between semi-classical string quantization and exact conformal field theory quantization of strings in 2+1 Anti de Sitter spacetime. More precisely, considering the WZWN model corresponding to $SL(2, R)$ and its covering group, we construct quantum *coherent* string states, which generalize the ordinary coherent states of quantum mechanics, and show that in the classical limit they correspond to oscillating circular strings. After quantization, the spectrum is found to consist of two parts: A continuous spectrum of low mass states (partly tachyonic) fulfilling the standard spin-level condition necessary for unitarity $|j| < k/2$, and a discrete spectrum of high mass states with asymptotic behaviour $m^2\alpha' \propto N^2$ (N positive integer). The quantization condition for the high mass states arises from the condition of finite positive norm of the coherent string states, and the result agrees with our previous results obtained using semi-classical quantization. In the $k \rightarrow \infty$ limit, all the usual properties of coherent or *quasi-classical* states are recovered.

It should be stressed that we consider the circular strings only for simplicity and clarity, and that our construction can easily be used for other string configurations too. We also compare our results with those obtained in the recent preprint hep-th/0001053 by Maldacena and Ooguri.

1 Introduction

The question about the spectrum of bosonic string theory in 3-dimensional Anti de Sitter space, $AdS_3 \cong SL(2, R) \cong SU(1, 1)$, attracted a lot of interest about 10 years ago [1-10], as a first example of exact string quantization on a manifold with curved space *and* curved time. It was immediately realized [1] that the problem of unitarity was much more complicated than in flat Minkowski spacetime [11, 12], in the sense that the Virasoro constraints for AdS_3 themselves did not eliminate all negative norm states [1]. However, in a series of papers, going back to [2, 3, 4] (see also [6-10, 13, 14]), it has been argued that unitarity can be ensured (at least for free strings) by imposing certain restrictions on the allowed representations, exemplified by the now well-known spin-level restriction for the discrete representations $|j| < k/2$, where j is the spin and k is the level of the $SL(2, R)$ WZWN model. For a somewhat different approach towards unitarity, see [15, 16].

More recently, the interest in AdS_3 (as well as higher dimensional AdS spaces) has increased in connection with the conjecture [17] relating supergravity and superstring theory on AdS space with a conformal field theory on the boundary. In such constructions, AdS_3 often appears on the 10-dimensional supergravity/superstring side in a cartesian product with some other compact spaces, for instance as $AdS_3 \times S^3 \times T^4$. Thus, again it has become extremely important to understand the precise spectrum of string theory in AdS_3 .

Even if the problem of unitarity apparently could be solved by the spin-level condition (although this question is certainly not completely settled yet), several problems remained. One of the most important being that the spin-level restriction together with the mass-shell condition imposes a restriction on the grade (to avoid confusion with the level k , we use the word "grade" for what is usually called level). This restriction on the grade means that for fixed level k , a string living in AdS_3 can only be excited to its very lowest grades. In other words, we are faced with the problem that it seems impossible to have very massive strings in AdS_3 .

On the other hand, the dynamics of classical strings and their semi-classical quantization in AdS_3 [18-22] does not seem to indicate any particular problems for very long and very massive strings, although the question of unitarity cannot really be addressed exactly in such studies. It is therefore highly interesting and important to understand how such long massive

strings can arise in an *exact* quantization scheme for strings in AdS_3 , without being in conflict with unitarity.

In this paper we suggest that long massive strings can be described as *coherent* string states based on one of the standard discrete representations of $SL(2, R)$. For simplicity and clarity, we shall construct quantum coherent string states corresponding, in the classical limit, to the circular strings discussed in [22], but our construction can be used for other string configurations too.

As for (most) other families of string states in AdS_3 , coherent string states generally do not have positive norm, even if they fulfil the Virasoro conditions. The condition of finite positive norm for the coherent states gives rise to certain restrictions on the spin j , which in turn restricts the mass of the states. We show that the finite positive norm condition for our coherent string states leads to a mass-spectrum consisting of two parts: A continuous spectrum of low mass states (partly tachyonic) where j fulfils the standard spin-level restriction, as well as an infinite tower of discrete high mass states for which the mass-formula is given by eq.(5.20) and asymptotically is $m^2\alpha' \propto N^2$ (N integer). This result agrees, to leading order, with what was found using semi-classical quantization [19, 20, 22].

When completing this paper, a recent preprint by Maldacena and Ooguri appeared [23], considering a similar problem. Our construction, however, is completely different from theirs. First of all, their massive strings are based on descendents of primary states for a new set of $SL(2, R)$ representations obtained from the standard ones by a "spectral flow" operation [9], whereas our massive strings are based on coherent states of descendents of primary states from the standard $SL(2, R)$ representations. Secondly, their construction relies heavily on the existence of some internal compact manifold \mathcal{M} , which is assumed to give a large contribution to the total world-sheet energy-momentum tensor, whereas our construction works directly for AdS_3 without need of any additional internal compact manifold (although of course we could easily include an internal compact manifold as well). In fact, with the internal compact manifold, the construction of [23] gives only a finite number of very massive states, and without the internal manifold it gives at most a few very massive states (or even none, depending on some other parameters). Our construction gives in any case an *infinite* tower of more and more massive states in the AdS_3 and $SL(2, R)$ background. This is again in agreement with the previous semi-classical quantization results [19, 20, 22] giving an

infinite number of string states in the AdS_3 and $SL(2, R)$ background. Yet another difference between the two approaches has to do with the world-sheet energy L_0 . In the construction of [23], L_0 is not bounded from below for the representations obtained by the spectral flow, while in our construction L_0 is bounded from below since we are using the standard representations. However, we are not working with eigenstates of L_0 thus the standard mass-shell condition $(L_0 - 1)|\psi\rangle = 0$ is replaced by $\langle\psi|(L_0 - 1)|\psi\rangle = 0$. In any case, the mass-shell condition eventually selects those states with " $L_0 = 1$ ".

It must be noticed however, that in despite of the differences between the construction of [23] and our construction, it turns out that the final results for the mass-spectrum (at least when an internal compact manifold with a large contribution to the world-sheet energy-momentum tensor is assumed) are more or less identical; namely, a low mass continuous spectrum and a high mass discrete spectrum, where the energy (or mass) to leading order grows linearly with an integer.

Interestingly enough, our quantum coherent states are a string generalization of the ordinary coherent states of quantum mechanics. All the usual properties of ordinary coherent states [24] are obtained in the $k \rightarrow \infty$ limit. For instance, the low mass continuous spectrum of string states become the ordinary coherent states, eigenstates of the annihilation operator, for any value of the spin $j \leq -1/2$, while the high mass discrete spectrum of string states completely disappears, pushed towards infinite mass. These are precisely the properties which in quantum mechanics characterise coherent states as *quasi-classical*, being the states for which quantum uncertainty is minimal.

Our paper is organized as follows. In Section 2, we review the classical $SL(2, R)$ WZWN model, mainly to set our conventions and normalizations. We also give a simple derivation of the reduction of the classical equations of motion to the Liouville equation [21, 25]. In Section 3, we reconsider the classical oscillating circular strings [22] in terms of $SL(2, R)$ currents. In Section 4, we present the standard results of the quantization of conformal field theories on a group manifold [28]; we only give the results which we will use later. In Section 5, we then turn to the construction of the quantum coherent string states. We derive the expression for the norm of such states and show that the condition of finite positive norm leads to a mass-spectrum as explained above. We also show that our coherent string states lead to non-vanishing expectation values only for the components of the currents

corresponding to the classical oscillating circular strings. Finally in Section 6, we have some concluding remarks.

2 $SL(2, \mathbb{R})$ WZWN Model. The Classical Picture.

Our starting point is the sigma-model action including the WZWN term at level k [26]

$$S = -\frac{k}{8\pi} \int_M d\tau d\sigma \eta^{\alpha\beta} \text{Tr}[g^{-1} \partial_\alpha g g^{-1} \partial_\beta g] - \frac{k}{12\pi} \int_B \text{Tr}[g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg] \quad (2.1)$$

Here M is the boundary of the manifold B , and g is a group-element of the group under consideration (later taken to be $SL(2, \mathbb{R})$). The classical string equations of motion are

$$\partial_- (g^{-1} \partial_+ g) = 0 \quad (2.2)$$

where we introduced world-sheet light-cone coordinates $\sigma^\pm = \tau \pm \sigma$. The world-sheet energy-momentum tensor is

$$T_{\pm\pm} = -\frac{2}{k} \text{Tr}(J_\pm J_\pm) \quad (2.3)$$

where the conserved currents, $\partial_\pm J_\mp = 0$, are given by

$$J_+ = ik g^{-1} (\partial_+ g), \quad J_- = -ik (\partial_- g) g^{-1} \quad (2.4)$$

and the string constraints are

$$\text{Tr}[(g^{-1} \partial_\pm g)(g^{-1} \partial_\pm g)] = 0 \quad (2.5)$$

Equation (2.2) is trivially solved by [26]

$$g(\sigma^+, \sigma^-) = g_R(\sigma^-) g_L(\sigma^+) \quad (2.6)$$

It follows that

$$J_+ = ik g_L^{-1} (\partial_+ g_L), \quad J_- = -ik (\partial_- g_R) g_R^{-1} \quad (2.7)$$

and the constraints, eq.(2.5), separate

$$\text{Tr}[(g_L^{-1}\partial_+g_L)^2] = \text{Tr}[(g_R^{-1}\partial_-g_R)^2] = 0 \quad (2.8)$$

In the case of $SL(2, R)$, the group elements are given by

$$g_L(\sigma^+) = \begin{pmatrix} \tilde{a}(\sigma^+) & \tilde{u}(\sigma^+) \\ -\tilde{v}(\sigma^+) & \tilde{b}(\sigma^+) \end{pmatrix}, \quad g_R(\sigma^-) = \begin{pmatrix} a(\sigma^-) & u(\sigma^-) \\ -v(\sigma^-) & b(\sigma^-) \end{pmatrix} \quad (2.9)$$

subject to the normalization conditions

$$\tilde{a}(\sigma^+)\tilde{b}(\sigma^+) + \tilde{u}(\sigma^+)\tilde{v}(\sigma^+) = a(\sigma^-)b(\sigma^-) + u(\sigma^-)v(\sigma^-) = 1 \quad (2.10)$$

Then the constraints, eqs.(2.8), are simply (from now on we do not write explicitly the arguments (σ^\pm) of the functions)

$$\tilde{a}_+\tilde{b}_+ + \tilde{u}_+\tilde{v}_+ = a_-b_- + u_-v_- = 0 \quad (2.11)$$

where we introduced the notation $a_- = \partial_-a$, $\tilde{a}_+ = \partial_+\tilde{a}$, etc.

As for the currents, it is convenient to make a Pauli decomposition

$$J_\pm = \eta_{ab}J_\pm^a t^b \quad (2.12)$$

in terms of the Pauli matrices,

$$t^1 = \frac{i}{2}\sigma^1, \quad t^2 = -\frac{i}{2}\sigma^3, \quad t^3 = \frac{1}{2}\sigma^2 \quad (2.13)$$

such that

$$\text{Tr}(t^a t^b) = -\frac{1}{2}\eta^{ab}, \quad [t^a, t^b] = i\epsilon^{abc}t_c \quad (2.14)$$

(Our conventions are: $\eta^{ab} = \text{diag}(1, 1, -1)$ and $\epsilon^{123} = +1$).

It is also standard to introduce

$$J_-^\pm = J_-^1 \pm iJ_-^2, \quad J_+^\pm = J_+^1 \pm iJ_+^2 \quad (2.15)$$

It is now straightforward to write down explicit expressions for the currents in terms of the group elements, eqs.(2.9),

$$\begin{aligned} J_-^\pm &= -k([au_- - ua_- + vb_- - bv_-] \pm 2i[ab_- + uv_-]) \\ J_-^3 &= k[vb_- - bv_- + ua_- - au_-] \\ J_+^\pm &= k([\tilde{b}\tilde{u}_+ - \tilde{u}\tilde{b}_+ + \tilde{v}\tilde{a}_+ - \tilde{a}\tilde{v}_+] \pm 2i[\tilde{v}\tilde{u}_+ + \tilde{a}\tilde{b}_+]) \\ J_+^3 &= -k[\tilde{v}\tilde{a}_+ - \tilde{a}\tilde{v}_+ + \tilde{u}\tilde{b}_+ - \tilde{b}\tilde{u}_+] \end{aligned} \quad (2.16)$$

Notice also that

$$T_{\pm\pm} = \frac{1}{k}(J_{\pm}^+ J_{\pm}^- - J_{\pm}^3 J_{\pm}^3) \quad (2.17)$$

such that the conditions $T_{\pm\pm} = 0$ again lead to eqs.(2.11), as they should.

We close this section with a few comments about the invariant string size and the reduction of the classical equations of motion to the Liouville equation [21] (for a review of different methods, see Ref.[25]).

The line-element on the group manifold is given by

$$dS^2 = \frac{1}{H^2} \text{Tr}[(g^{-1}dg)^2] \quad (2.18)$$

where H^{-1} is the length-scale, which up to a numerical factor is related to k and α' by [26, 27]

$$k = \frac{1}{H^2 \alpha'} \quad (2.19)$$

where α' is related to the string tension T in the usual way, $T = (2\pi\alpha')^{-1}$.

The line-element on the group manifold induces the following proper line-element on the string world-sheet

$$ds^2 = -\frac{e^\alpha}{2H^2} d\sigma^+ d\sigma^- \quad (2.20)$$

Here $\alpha = \alpha(\sigma^+, \sigma^-)$ is the fundamental quadratic form, which determines the invariant string size, and is defined by

$$e^\alpha \equiv -\text{Tr}[(g^{-1}\partial_+g)(g^{-1}\partial_-g)] \quad (2.21)$$

In the case of $SL(2, R)$, one finds

$$\begin{aligned} e^\alpha &= [av_- - va_-][\tilde{a}\tilde{u}_+ - \tilde{u}\tilde{a}_+] + [ub_- - bu_-][\tilde{v}\tilde{b}_+ - \tilde{b}\tilde{v}_+] \\ &+ 2[ab_- + vu_-][\tilde{b}\tilde{a}_+ + \tilde{v}\tilde{u}_+] \end{aligned} \quad (2.22)$$

By differentiating this identity twice and by using some simple algebra, we get the equation

$$\alpha_{+-} = 2f(\sigma^-)\tilde{g}(\sigma^+)e^{-\alpha} \quad (2.23)$$

where the functions $f = f(\sigma^-)$ and $\tilde{g} = \tilde{g}(\sigma^+)$ are given by

$$f = \frac{1}{2} \left(\frac{u_-}{a_-} [av_{--} - va_{--}] - \frac{v_-}{b_-} [ub_{--} - bu_{--}] \right) \quad (2.24)$$

$$\tilde{g} = \frac{1}{2} \left(\frac{\tilde{u}_+}{\tilde{b}_+} [\tilde{v}\tilde{b}_{++} - \tilde{b}\tilde{v}_{++}] - \frac{\tilde{v}_+}{\tilde{a}_+} [\tilde{a}\tilde{u}_{++} - \tilde{u}\tilde{a}_{++}] \right) \quad (2.25)$$

The product $f(\sigma^-)\tilde{g}(\sigma^+)$ in eq.(2.23) can be absorbed by a conformal transformation, thus we conclude that the most general equation fulfilled by the fundamental quadratic form α is

$$\alpha_{+-} + Ke^{-\alpha} = 0, \quad (2.26)$$

where:

$$K = \begin{cases} +1, & f(\sigma^-)\tilde{g}(\sigma^+) < 0 \\ -1, & f(\sigma^-)\tilde{g}(\sigma^+) > 0 \\ 0, & f(\sigma^-)\tilde{g}(\sigma^+) = 0 \end{cases} \quad (2.27)$$

Equation (2.26) is either the Liouville equation ($K = \pm 1$), or the free wave equation ($K = 0$). This result was obtained in a different way and discussed in detail in Ref. [21].

3 Circular Strings

Circular strings on the $SL(2, R)$ group manifold were considered in detail in Ref. [22]. In this section we translate the results into the formalism of $SL(2, R)$ currents.

Circular strings are most easily discussed using a different parametrization of $SL(2, R)$, corresponding to the static coordinates for Anti de Sitter spacetime. We first write the $SL(2, R)$ group-element in the form

$$g = g_R g_L = \begin{pmatrix} A & U \\ -V & B \end{pmatrix} = \begin{pmatrix} a\tilde{a} - u\tilde{v} & a\tilde{u} + u\tilde{b} \\ -v\tilde{a} - b\tilde{v} & b\tilde{b} - v\tilde{u} \end{pmatrix} \quad (3.1)$$

and then introduce coordinates (t, r, ϕ) by

$$\begin{aligned} A &= \sqrt{1 + H^2 r^2} \cos(Ht) + Hr \cos(\phi) \\ B &= \sqrt{1 + H^2 r^2} \cos(Ht) - Hr \cos(\phi) \\ U &= \sqrt{1 + H^2 r^2} \sin(Ht) - Hr \sin(\phi) \\ V &= \sqrt{1 + H^2 r^2} \sin(Ht) + Hr \sin(\phi) \end{aligned} \quad (3.2)$$

In these coordinates, the line-element (2.18) on the group manifold becomes

$$dS^2 = -(1 + H^2 r^2)dt^2 + \frac{dr^2}{1 + H^2 r^2} + r^2 d\phi^2 \quad (3.3)$$

i.e., the standard parametrization of $2 + 1$ dimensional Anti de Sitter space-time using static coordinates [29]. As usual we unwind the temporal coordinate t , corresponding to considering the covering group of $SL(2, R)$. Moreover, the anti-symmetric tensor which can be read off from eq.(2.1), is given by

$$B_{t\phi} = -B_{\phi t} = \frac{1}{2} H r^2 \quad (3.4)$$

with all other components vanishing.

In the (t, r, ϕ) coordinates, the oscillating circular strings are given by [22]

$$\begin{aligned} \phi &= \sigma \\ Ht &= \arctan\left(\frac{1 + EH}{\sqrt{1 + 2EH}} \tan(\sqrt{1 + 2EH} \tau)\right) - \tau \\ r &= \frac{E}{\sqrt{1 + 2EH}} \sin(\sqrt{1 + 2EH} \tau) \end{aligned} \quad (3.5)$$

where E is an integration constant.

It is now straightforward to work backwards and read off the explicit expressions for the leftmoving and rightmoving group-elements, eqs.(2.9). The expressions are however not very enlightening, so we give them in Appendix A. It is more interesting to consider directly the leftmoving and rightmoving currents, eqs.(2.16). After some simple algebra, they are found to be

$$\begin{aligned} J_-^\pm &= \pm i E H k e^{\pm i \sigma^-} \\ J_-^3 &= -E H k \\ J_+^\pm &= \mp i E H k e^{\mp i \sigma^+} \\ J_+^3 &= E H k \end{aligned} \quad (3.6)$$

Thus, the circular strings contain only modes corresponding to $n = 0$ and $n = \pm 1$. This was of course to be expected for a circular string, c.f. circular strings in Minkowski spacetime, but it is in fact highly implicit in the parametrization (3.5). From the conformal field theory point of view, the parametrization (3.6) is thus more natural.

4 Quantization

In this section we give a short review of quantization of conformal field theories, corresponding to bosonic strings on group manifolds (see for instance

Refs. [28, 13]). This is mainly to fix our conventions and normalizations. We only give the results which we shall use in Section 5.

The currents J_{\pm}^a , as introduced in eq.(2.12), can be expanded in Fourier series

$$J_{-}^a = \sum_{n=-\infty}^{\infty} J_n^a e^{-in\sigma^{-}}; \quad (J_n^a)^{\dagger} = J_{-n}^a \quad (4.1)$$

and similarly for J_{+}^a in terms of σ^{+} . In the following we shall consider only the rightmovers $(-)$; the construction for the leftmovers $(+)$ is of course similar. For simplicity we shall therefore also skip the minus indices of J_{-}^a , T_{--} etc. The $SL(2, R)$ Kac-Moody algebra is

$$[J_m^a, J_n^b] = i\epsilon^{ab} {}_c J_{m+n}^c + \frac{k}{2} m \eta^{ab} \delta_{n+m} \quad (4.2)$$

In terms of the currents, eq.(2.15), the algebra becomes

$$\begin{aligned} [J_m^{+}, J_n^{-}] &= -2J_{m+n}^3 + km\delta_{m+n} \\ [J_m^3, J_n^{\pm}] &= \pm J_{m+n}^{\pm} \\ [J_m^3, J_n^3] &= -\frac{k}{2} m \delta_{m+n} \end{aligned} \quad (4.3)$$

At the quantum level, the world-sheet energy-momentum tensor takes the Sugawara form

$$T = \frac{1}{k-2} \eta_{ab} : J^a J^b : = \frac{1}{k-2} : (J^{+} J^{-} - J^3 J^3) : \quad (4.4)$$

Its Fourier modes

$$T = \sum_{n=-\infty}^{\infty} L_n e^{-in\sigma^{-}} \quad (4.5)$$

are given by

$$L_n = \frac{1}{k-2} \sum_{l=-\infty}^{\infty} : \left(\frac{1}{2} (J_{n-l}^{+} J_l^{-} + J_{n-l}^{-} J_l^{+}) - J_{n-l}^3 J_l^3 \right) : \quad (4.6)$$

They fulfill the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n} \quad (4.7)$$

where the central charge is given by

$$c = \frac{3k}{k-2} \quad (4.8)$$

Demanding $c = 26$, corresponding to conformal invariance, gives $k = 52/23$. Notice also the commutators

$$[L_n, J_m^\pm] = -m J_{n+m}^\pm, \quad [L_n, J_m^3] = -m J_{n+m}^3 \quad (4.9)$$

which will be useful in the following.

The Kac-Moody algebra contains the subalgebra of zero modes J_0^a , for which the quadratic Casimir is

$$Q = \eta_{ab} J_0^a J_0^b = \frac{1}{2} (J_0^+ J_0^- + J_0^- J_0^+) - J_0^3 J_0^3 \quad (4.10)$$

The primary states, which are quantum states $|jm\rangle$ at grade zero ("base-states" or "ground-states"), are characterised by

$$Q|jm\rangle = -j(j+1)|jm\rangle, \quad J_0^3|jm\rangle = m|jm\rangle \quad (4.11)$$

Moreover, they fulfill

$$J_0^\pm|jm\rangle = \sqrt{m(m \pm 1) - j(j+1)} |jm \pm 1\rangle \quad (4.12)$$

as well as

$$J_l^a|jm\rangle = 0; \quad l > 0 \quad (4.13)$$

The primary states must belong to one of the unitary representations of $SL(2, R)$ (or its covering group) [30, 2]. We shall return to this point in the next section.

From the primary states, one can construct the excited states as descendants by applying J_{-l}^a operators (l is a positive integer)

$$|\psi\rangle = J_{-l_1}^{a_1} J_{-l_2}^{a_2} \dots J_{-l_r}^{a_r} |jm\rangle \quad (4.14)$$

and so on. The physical state conditions (the mass-shell condition and the Virasoro primary condition) are then

$$(L_0 - 1)|\psi\rangle = 0 \quad (4.15)$$

$$L_l|\psi\rangle = 0; \quad l > 0 \quad (4.16)$$

For a physical state of the form (4.14) at grade $n = \sum l_i$, the mass-shell condition gives

$$n - \frac{j(j+1)}{k-2} = 1 \quad (4.17)$$

Identifying (minus) the quadratic Casimir, eq.(4.10), of the base-state with the mass-squared [5], or more precisely we normalize the mass as

$$m^2\alpha' \equiv \frac{j(j+1)}{k-2} \quad (4.18)$$

we then see that the mass-squared grows linearly with the grade

$$m^2\alpha' = n - 1 \quad (4.19)$$

This is just like for strings in flat Minkowski spacetime.

5 Coherent String States

The idea is now to construct exact quantum states with properties similar to the classical circular strings considered in Section 3. More precisely, our aim is to construct states $|\psi\rangle$ such that the only components of the currents giving non-vanishing expectation values, $\langle \psi | J^a | \psi \rangle \neq 0$, are those components corresponding to the non-vanishing classical currents, eq.(3.6). In other words, considering for simplicity only the rightmovers, then only the components J_{-1}^+ , J_{+1}^- and J_0^3 should have non-vanishing expectation values; all other components J_n^+ ($n \neq -1$), J_n^- ($n \neq +1$) and J_n^3 ($n \neq 0$) must have zero expectation values.

Several problems immediately appear: If we consider states of the form (4.14), it would be possible to obtain a non-vanishing expectation value for J_0^3 , but it would certainly be impossible to get non-vanishing expectation values of J_{-1}^+ and J_{+1}^- . Moreover, as mentioned at the end of Section 4, states of the form (4.14) give rise to a mass-spectrum where the mass-squared grows linearly with the grade as in flat space. However, semi-classical quantization of the circular strings has been shown [22] to lead to a mass-spectrum where $m^2\alpha' \propto N^2$ (N positive integer), at least for the high mass states. Fortunately, it turns out that both problems can be solved by considering *coherent*

string states on the $SL(2, R)$ group manifold (a similar construction for the $SU(2)$ group manifold was considered in [31]).

As a starting point, we consider states of the form

$$\left(J_{-1}^+\right)^n |jj>; \quad n \geq 0 \quad (5.1)$$

where $|jj>$ belongs to the highest weight discrete series D_j^- [30, 2], with states $|jm>$

$$j \leq -1/2, \quad m = j, j-1, \dots \quad (5.2)$$

Since we shall consider the covering group of $SL(2, R)$, there are no further restrictions on j , i.e., it need not be integer or half-integer [30, 2]. In particular, from eq.(4.12) it follows that

$$J_0^+ |jj> = 0, \quad J_0^- |jj> = \sqrt{-2j} |jj-1> \quad (5.3)$$

The states eq.(5.1) fulfill the Virasoro primary condition

$$L_l \left(J_{-1}^+\right) |jj> = 0; \quad l > 0 \quad (5.4)$$

and the mass-shell condition (4.15) leads to

$$n = 1 + \frac{j(j+1)}{k-2} \Leftrightarrow j = -\frac{1}{2} - \sqrt{(k-2)(n-1) + 1/4} \quad (5.5)$$

However, these states generally do not have positive norm. Indeed

$$\langle jj| \left(J_{+1}^-\right)^m \left(J_{-1}^+\right)^n |jj> = n! \delta_{nm} \prod_{i=1}^n \left(k-2+i-\sqrt{4(k-2)(n-1)+1}\right) \quad (5.6)$$

and the right hand side is generally not positive. For example, it is negative for $n = m = 2$ and $n = m = 3$ (using that $k = 52/23$). This is of course just a simple example illustrating the well known unitarity problem for strings on $SL(2, R)$ [1-10, 13] (for recent reviews, see [32, 33]).

We consider instead coherent states built from states of the form (5.1)

$$e^{\mu J_{-1}^+} |jj> \quad (5.7)$$

where μ is an arbitrary complex number. These states certainly also fulfill the Virasoro primary condition but, being coherent states, they obviously

are eigenstates of neither the number operator nor of the L_0 operator. We shall therefore impose a "weak" mass-shell condition

$$\langle jj|e^{\mu^* J_{+1}^-} (L_0 - 1) e^{\mu J_{-1}^+}|jj\rangle = 0 \quad (5.8)$$

Before evaluating the left hand side of eq.(5.8), it is necessary to consider the normalization of the states (5.7).

In ordinary quantum mechanics with creation and annihilation operators a^\dagger and a , respectively, and a vacuum state $|0\rangle$

$$[a, a^\dagger] = 1, \quad a|0\rangle = 0 \quad (5.9)$$

the excited states are constructed as

$$(a^\dagger)^n |0\rangle = \sqrt{n!} |n\rangle, \quad \langle n|m\rangle = \delta_{nm} \quad (5.10)$$

In that case, a coherent state can always be normalized. In fact, the coherent state

$$|\mu\rangle \equiv e^{-\frac{1}{2}\mu^*\mu} e^{\mu a^\dagger} |0\rangle \quad (5.11)$$

has unit norm, for arbitrary complex number μ . Notice also that the coherent state is an eigenstate of the annihilation operator

$$a|\mu\rangle = \mu|\mu\rangle \quad (5.12)$$

which can be taken as the definition of coherent states in ordinary quantum mechanics. (For more discussion, see for instance [24]).

In our case, the situation is somewhat different since we have a Kac-Moody algebra (4.2) with a non-Abelian term in the current algebra, and in particular since the group manifold $SL(2, R)$ is non-compact and has a time-like direction (contrary to the case of $SU(2)$ [31]). It implies that the coherent state (5.7) is not an eigenstate of the annihilation operator J_{+1}^-

$$J_{+1}^- e^{\mu J_{-1}^+} |jj\rangle = \mu (2j + k + \mu J_{-1}^+) e^{\mu J_{-1}^+} |jj\rangle \quad (5.13)$$

Moreover, the coherent state (5.7) can not be normalized for arbitrary complex number μ . One finds

$$\langle jj|e^{\mu^* J_{+1}^-} e^{\mu J_{-1}^+}|jj\rangle = 1 + \sum_{n=1}^{\infty} \frac{(\mu^* \mu)^n}{n!} \prod_{l=1}^n (2j + k - 1 + l) \quad (5.14)$$

The product on the right hand side goes as $n!$. Thus the infinite sum is convergent only if $\mu^*\mu < 1$, or if the infinite sum terminates after a finite number of terms (this happens if $2j + k - 1 + l = 0$, for some l). More precisely, the right hand side of eq.(5.14) is a finite positive number in the following two cases

(I): $\mu^*\mu < 1$ and j arbitrary ($j \leq -1/2$).

In this case the normalized state is

$$|\mu I\rangle = (1 - \mu^*\mu)^{j+k/2} e^{\mu J_{-1}^+} |jj\rangle \quad (5.15)$$

(II): $\mu^*\mu > 1$ and $j = -N - k/2$ ($N = 0, 1, 2, \dots$).

In this case the normalized state is

$$|\mu II\rangle = (\mu^*\mu - 1)^{-N} e^{\mu J_{-1}^+} | -N - k/2, -N - k/2 \rangle \quad (5.16)$$

Let us now return to the mass-shell condition, eq.(5.8), which gives rise to some additional constraints on μ and j . In the two cases, respectively, one finds

(I):

$$\mu^*\mu = \frac{1 + \frac{j(j+1)}{k-2}}{2j + k + 1 + \frac{j(j+1)}{k-2}} < 1 ; \quad -\frac{k}{2} < j \leq -\frac{1}{2} \quad (5.17)$$

(II):

$$\mu^*\mu = \frac{1 + \frac{j(j+1)}{k-2}}{2j + k + 1 + \frac{j(j+1)}{k-2}} > 1 ; \quad j = -N - \frac{k}{2} \quad (N = 1, 2, \dots) \quad (5.18)$$

Thus, the spectrum consists of two parts: **(I)** A continuous spectrum where j fulfills the standard spin-level condition [2-4, 6-10, 13, 14] $-k/2 < j \leq -1/2$, and **(II)** a discrete spectrum where j fulfills $j = -N - k/2$ (N positive integer). If we were considering ordinary descendent states of the form (4.14), we would generally not be allowed to have $|j| > k/2$ because of unitarity, thus the discrete part **(II)** would not be allowed. For the coherent states

under consideration here, there is however no problem since the quantization condition $j = -N - k/2$ (N positive integer) precisely ensures that they are all positive norm states.

Introducing the mass with the normalization as in eq.(4.18), we find that the continuous part of the spectrum **(I)** corresponds to

$$m^2\alpha' \in [\frac{-1}{4(k-2)}, \frac{k}{4}[= [-\frac{23}{24}, \frac{13}{23}[\quad (5.19)$$

where the last equality was obtained using $k = 52/23$, corresponding to conformal invariance. That is, the continuous part of the spectrum **(I)** consists of low mass states and is partly tachyonic.

On the other hand, the discrete spectrum **(II)** gives

$$m^2\alpha' = \frac{N^2}{k-2} \left(1 + \frac{k-1}{N} + \frac{k(k-2)}{4N^2} \right) \quad (5.20)$$

i.e., for the discrete part of the spectrum we find that $m^2\alpha' \propto N^2$ (N positive integer). Asymptotically, this is precisely what was found using semi-classical quantization [19, 20, 22]. It should be stressed, however, that N is not the eigenvalue of the number operator; as already mentioned, since we are working with coherent states, we do not have any eigenstates of the number operator. Thus N is simply a positive integer here.

Notice also that k , α' and the length-scale H in the quantum theory are related as in eq.(2.19), but with k replaced by $k-2$. With the present normalizations we therefore have exactly the same leading order behaviour, including the numerical coefficient, for the mass-squared as obtained using semi-classical quantization in Ref. [22].

Finally, the relation with the classical circular strings is established by noting that only the expectation values of J_{-1}^+ , J_{+1}^- and J_0^3 are non-vanishing ($i = I, II$)

$$\begin{aligned} \langle \mu i | J_l^+ | \mu i \rangle &= \begin{cases} (2j+k) \frac{\mu^*}{1-\mu^*\mu}, & l = -1 \\ 0, & l \neq -1 \end{cases} \\ \langle \mu i | J_l^- | \mu i \rangle &= \begin{cases} (2j+k) \frac{\mu}{1-\mu^*\mu}, & l = +1 \\ 0, & l \neq +1 \end{cases} \\ \langle \mu i | J_l^3 | \mu i \rangle &= \begin{cases} j + (2j+k) \frac{\mu^*\mu}{1-\mu^*\mu}, & l = 0 \\ 0, & l \neq 0 \end{cases} \end{aligned} \quad (5.21)$$

valid for both $|\mu I >$ and $|\mu II >$ for the respective values of μ and j , as given in eqs.(5.17)-(5.18). To obtain these results, as well as most other results in this section, we used the commutators listed in Appendix B.

6 Conclusion

We have shown that very massive string states in the $SL(2, R)$ WZWN model (corresponding to AdS_3) can be described as *coherent* states based on the standard discrete representation D_j^- . The spectrum of such states was shown to consist of two parts: A continuous low mass (partly tachyonic) part and a discrete high mass part. For the continuous part, the spin j must fulfill the standard spin-level restriction $-k/2 < j \leq -1/2$, while for the discrete part we get the quantization condition $j = -N - k/2$ (N positive integer). Although the latter seems to be in contradiction with the spin-level restriction, the quantization condition precisely ensures that *all* our coherent states have finite positive norm. Thus, no ghost-states are included in the spectrum. The mass spectrum of the discrete part of the spectrum, eq.(5.20), shows the asymptotic behaviour $m^2 \alpha' \propto N^2$. This is in precise agreement with our previous results obtained using semi-classical quantization [19, 20, 22]. The same asymptotic behaviour was also obtained in the recent preprint [23], although the construction there is completely different from ours, as discussed in more detail in the introduction.

In this paper we used, for simplicity and clarity, the example of an oscillating circular string. That is, our quantum coherent string states were constructed to lead to non-vanishing expectation values for very specific components of the currents, eq.(5.21). It is however easy to generalize our construction to other string configurations too.

We saw in Section 5 that the coherent states, eq.(5.7), do not have all the same properties of standard quantum mechanical coherent states [24]. For instance, they are not eigenstates of the annihilation operator. All the standard properties are however obtained in the following manner: First we must renormalize the currents $J_n^a \rightarrow \sqrt{k} J_n^a$, as follows from the algebra, eq.(4.3) (although this does not work for the zero-modes [5]). Secondly, for the coherent states we must renormalize the complex number μ by $\mu \rightarrow \mu/\sqrt{k}$, as follows from eq.(5.7). Finally, we let $k \rightarrow \infty$. Then, it follows that the non-Abelian piece drops out in eq.(5.13) and we get an eigenstate of the renor-

malized annihilation operator. More generally, by the same prescription we recover all the usual well-known properties of standard quantum mechanical coherent states [24]. For instance, the continuous spectrum, represented by the states (5.15), will now be valid for any $j \leq -1/2$, and the states become

$$|\mu I \rangle \rightarrow e^{-\frac{1}{2}\mu^*\mu} e^{\mu J_{-1}^+} |jj \rangle; \quad k \rightarrow \infty \quad (6.1)$$

c.f. eq.(5.11). On the other hand, the discrete part of the spectrum, represented by the states (5.16), disappears since all the states are pushed to infinite mass. These are precisely the properties which characterise the usual coherent states of quantum mechanics as *quasi-classical* states, for which the quantum uncertainty is minimal.

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7 Appendix A

In this appendix we give the explicit expressions for the group-elements (2.9) corresponding to the circular strings (3.5). For simplicity, we only give the results for the rightmovers.

$$\begin{aligned} a(\sigma^-) &= \frac{1}{\sqrt{1+2EH}} \sin\left(\sqrt{1+2EH} \frac{\sigma^-}{2}\right) \left[(1+EH) \sin\left(\frac{\sigma^-}{2}\right) + EH \cos\left(\frac{\sigma^-}{2}\right) \right] \\ &+ \cos\left(\frac{\sigma^-}{2}\right) \cos\left(\sqrt{1+2EH} \frac{\sigma^-}{2}\right) \end{aligned} \quad (7.1)$$

$$\begin{aligned} b(\sigma^-) &= \frac{1}{\sqrt{1+2EH}} \sin\left(\sqrt{1+2EH} \frac{\sigma^-}{2}\right) \left[(1+EH) \sin\left(\frac{\sigma^-}{2}\right) - EH \cos\left(\frac{\sigma^-}{2}\right) \right] \\ &+ \cos\left(\frac{\sigma^-}{2}\right) \cos\left(\sqrt{1+2EH} \frac{\sigma^-}{2}\right) \end{aligned} \quad (7.2)$$

$$u(\sigma^-) = \frac{1}{\sqrt{1+2EH}} \sin\left(\sqrt{1+2EH} \frac{\sigma^-}{2}\right) \left[(1+EH) \cos\left(\frac{\sigma^-}{2}\right) + EH \sin\left(\frac{\sigma^-}{2}\right) \right]$$

$$- \sin\left(\frac{\sigma^-}{2}\right) \cos\left(\sqrt{1+2EH} \frac{\sigma^-}{2}\right) \quad (7.3)$$

$$\begin{aligned} v(\sigma^-) &= \frac{1}{\sqrt{1+2EH}} \sin\left(\sqrt{1+2EH} \frac{\sigma^-}{2}\right) \left[(1+EH) \cos\left(\frac{\sigma^-}{2}\right) - EH \sin\left(\frac{\sigma^-}{2}\right) \right] \\ &- \sin\left(\frac{\sigma^-}{2}\right) \cos\left(\sqrt{1+2EH} \frac{\sigma^-}{2}\right) \end{aligned} \quad (7.4)$$

Now using eq.(2.16), it is straightforward to obtain (3.6) for the rightmovers. The derivation for the leftmovers is similar.

8 Appendix B

In this appendix we list some useful commutators used in Section 5.

$$[J_m^3, (J_{-1}^+)^n] = n J_{m-1}^+ (J_{-1}^+)^{n-1} \quad (8.1)$$

$$[J_m^+, (J_{+1}^-)^n] = -n (2J_{m+1}^3 - km\delta_{m+1}) (J_{+1}^-)^{n-1} - n(n-1) J_{m+2}^- (J_{+1}^-)^{n-2} \quad (8.2)$$

$$[J_m^-, (J_{-1}^+)^n] = n (J_{-1}^+)^{n-1} (2J_{m-1}^3 + km\delta_{m-1}) + n(n-1) (J_{-1}^+)^{n-2} J_{m-2}^+ \quad (8.3)$$

These identities are most easily proved by induction using eqs.(4.3). It follows that

$$[J_m^3, e^{\mu J_{-1}^+}] = \mu J_{m-1}^+ e^{\mu J_{-1}^+} \quad (8.4)$$

$$[J_m^+, e^{\mu^* J_{+1}^-}] = -\mu^* (2J_{m+1}^3 - km\delta_{m+1} + \mu^* J_{m+2}^-) e^{\mu^* J_{+1}^-} \quad (8.5)$$

$$[J_m^-, e^{\mu J_{-1}^+}] = \mu e^{\mu J_{-1}^+} (2J_{m-1}^3 + km\delta_{m-1} + \mu J_{m-2}^+) \quad (8.6)$$

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